I SURDS AND INDICES

Positive Integral Power : For any real number 'a' and a positive integer 'n', we define a^n as $a^n = a \times a \times a \times$ ...... $\times a$ (n factors) we have specific terms for $n = 2$ called **square** and for $n = 3$ called **cube** . For example,

(i) $5^2 = 5 \times 5 = 25$

(ii) $\left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{27}{8}$

For any non - zero real number a, we define $a^\circ = 1$.

Thus, $5^\circ = 1$, $\left(\frac{5}{9}\right)^\circ = 1$

Negative Integral Power : For any non - zero real number 'a' and positive integer 'n', we define $a^{-n} = \frac{1}{a^n}$

For example,

$(3)^{-2} = \frac{1}{3^2} = \frac{1}{3 \times 3} = \frac{1}{9}$

LAWS OF INDICES

First law : If 'a' is any real number and m, n are positive integers, then $a^m \times a^n = a^{m+n}$

Conversely, $a^{m+n} = a^m \times a^n$

For example, $2^5 \times 2^3 = 2^{5+3} = 2^8$

Second law : If a is a non-zero number and m, n are positive integers, then, $\frac{a^m}{a^n} = a^{m-n}$

For example,

(i) $3^5 \div 3^2 = 3^{5-2} = 3^3 = 27$

(ii) $\left(\frac{3}{7}\right)^5 \div \left(\frac{3}{7}\right)^3 = \left(\frac{3}{7}\right)^{5-3} = \left(\frac{3}{7}\right)^2$

Third law : If 'a' is any real number and m, n are positive integers, then $(a^m)^n = a^{mn} = (a^n)^m$

For example,

(i) $(2^2)^5 = 2^{2\times5} = 2^{10}$

(ii) $\{(3^4)^3\}^5 = 2^{4\times3\times5} = 3^{60}$

Fourth law : If a, b are real numbers and m, n are positive integers then.

(i) $(ab)^n = a^n b^n$

(ii) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq o$

For example,

(i) $6^5 = (2\times3)^5 = 2^5 \times 3^5$

(ii) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

Principal nth Root of a Positive Real Number : if 'a' is a positive real number and n is a positive integer, then the principal nth root of 'a' is the unique positive real number x such that $x^n = a$. The principal nth root of a positive real number a is denoted by $(a)^{\frac{1}{n}}$ of $\sqrt[n]{a}$.

For example,

(i) $(9)^{\frac{1}{2}} = 3$ because $3^2 = 9$

(ii) $(64)^{\frac{1}{3}} = 4$ because $4^3 = 64$

Principal nth Root of a Negative Real Number : If 'a' is a negative real number and 'n' is an odd positive integer, then the principal nth root of 'a' is defined as $-|a|^{\frac{1}{n}}$.

For example, $(-8)^{\frac{1}{3}} = -8^{\frac{1}{3}} = -2$

If 'a' is a negative real number and n is an even integer, then the principal nth root of 'a' is not defined. Therefore, $(-16)^{\frac{1}{2}}$ is a meaningless quantity, if we confine ourselves to the set of real numbers. 9

Surds : If 'a' is a rational number and n is a positive integer such that the nth root of 'a' i.e. $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called a surd. In other words, an irrational root of a rational number is called a surd.

For the surd $\sqrt[n]{a}$, n is called order of the surd and a is called the radicand. The symbol '$\sqrt{\ }$' is called the radical sign.

LAWS OF SURDS

First law: For any positive integer 'n' and a positive rational number 'a' $\left(\sqrt[n]{a}\right)^n = a$

Second law : If n is positive integer and a,b are rational numbers, then $\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$

Third law If n is a positive integer and a,b are rational numbers, then, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Fourth law : If m, n are positive integers and 'a' is positive rational number, then $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

Fifth law : If m, n are positive integers and 'a' is a positive rational number, then

$$\sqrt[n]{\sqrt[m]{\left(a^p\right)^m}} = \sqrt[n]{a^p} = \sqrt[mn]{a^{pm}}$$

Comparison of Surds : To compare the magnitude of surds of different orders, we change them into the surds of the same order. This order is the L.C.M. of the orders of the given surds. Then, two surds of the same order can be easily compared by just comparing their radicands. Following example well illustrate the procedure.

1. Arrange the following surds in ascending order of their magnitude.

$\sqrt{5}$, $\sqrt[3]{9}$, $\sqrt[6]{105}$

Solution : The given surds are $\sqrt{5}$, $\sqrt[3]{9}$, $\sqrt[6]{105}$.

The order of these surds are 2, 3 and 6 respectively.

L.C.M. of 2, 3, 6 is 6 .

So, we convert each surd into a surd of order 6 .

Now, $\sqrt{5} = (5)^{\frac{1}{2}} = (5)^{\frac{1\times3}{2\times3}} = 5^{\frac{3}{6}} = \left(5^3\right)^{\frac{1}{6}}$

$= (125)^{\frac{1}{6}} = \sqrt[6]{125}$

Or, we can write $\sqrt{5} = \sqrt[6]{5^3} = \sqrt[6]{125}$

$\sqrt[3]{9} = \sqrt[6]{9^2} = \sqrt[6]{81}$, $\sqrt[6]{105}$ is already a surd of order 6.

Since $81 < 105 < 125$

$\Rightarrow \sqrt[3]{9} < \sqrt[6]{105} < \sqrt{5}$

KINDS OF SURDS

Monomial surd : A surd consisting of one and only one term is called a monomial surd.

For example, $\sqrt{3}$, $\sqrt{2}$, are monomial surds.

Binomial surd : An expression consisting of the sum or difference of two monomial surds or a monomial surd and a rational number is called a binomial surd.

For example, $\sqrt{3} + \sqrt{7}$, $5 + \sqrt{3}$ etc. are binomial surds.

Trinomial surd : An expression consisting of three terms at least two of which are monomial surds, is called a trinomial surd.

For example, $\sqrt{3} + \sqrt{7} - \sqrt{8}$,

$\sqrt{5} + \sqrt{3} + 7$ are trinomial surds.

Rationalisation of surds : When the product of two surds is a rational number, then each of them is called the Rationalising Factor of the other.

For example, $3\sqrt{5} \times \sqrt{5} = 15$,

There fore, $\sqrt{5}$ is a rationalising factor of $3\sqrt{5}$. In a binomial surd of the form $\sqrt{a} \pm \sqrt{b}$, the rationalising factors are $\sqrt{a} \mp \sqrt{b}$

Let's consider some examples asked in previous

1. Simplify: $\sqrt{\dfrac{19+8\sqrt{3}}{7-4\sqrt{3}}}$

[SSC Graduate Level (Asstt. Grade) Main exam: 03.07.2004]

Solution : Expression $= \sqrt{\dfrac{19+8\sqrt{3}}{7-4\sqrt{3}}}$

$= \sqrt{\dfrac{19+2\times4\times\sqrt{3}}{7-2\times2\times\sqrt{3}}} = \sqrt{\dfrac{16+3\times2\times4\times\sqrt{3}}{4+3-2\times2\times\sqrt{3}}}$

$= \sqrt{\dfrac{(4)^2+\left(\sqrt{3}\right)^2+2\times4\times\sqrt{3}}{(2)^2+\left(\sqrt{3}\right)^2-2\times2\times\sqrt{3}}} = \sqrt{\dfrac{\left(4+\sqrt{3}\right)^2}{\left(2-\sqrt{3}\right)^2}} = \dfrac{4+\sqrt{3}}{2-\sqrt{3}}$

On rationalising the denominator we get,

$= \dfrac{\left(4+\sqrt{3}\right)\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)} = \dfrac{8+2\sqrt{3}+4\sqrt{3}+3}{4-3}$

$= 11+6\sqrt{3}$

2. Simplify :

$$\sqrt{\frac{(2.4)^6+9(5.76)+6(2.4)^4}{(2.4)^2+6(5.76)+9}} + \frac{\left[\left(3^{-2}\right)^{-5}\right]^{\frac{1}{5}} + \left[\left(4^{-3}\right)^{-6}\right]^{\frac{1}{6}} - \left(3^{-4}\right)^{\frac{-1}{2}}}{\left[\left(2^{-3}\right)^{-4}\right]^{\frac{1}{4}}}$$

[SSC Graduate Level (UDC) Main Exam; 27.06.2004]

Solution : Let part I $= \sqrt{\dfrac{(2.4)^6+9(5.76)+6(2.4)^4}{(2.4)^4+6(5.76)+9}}$

$= \sqrt{\dfrac{(2.4)^6+9(2.4)^2+6(2.4)^4}{(2.4)^4+6(2.4)^2+9}}$

$= \sqrt{\dfrac{(2.4)^2\left[(2.4)^4+9+6(2.4)^2\right]}{(2.4)^4+6(2.4)^2+9}} = \sqrt{(2.4)^2} = 2.4$

Part II $= \dfrac{\left[\left(3^{-2}\right)^{-5}\right]^{\frac{1}{5}} + \left[\left(4^{-3}\right)^{-6}\right]^{\frac{1}{6}} - \left(3^{-4}\right)^{\frac{-1}{2}}}{\left[\left(2^{-3}\right)^{-4}\right]^{\frac{1}{4}}}$

$$= \frac{3^{(-2)\times(-5)\times\frac{1}{5}} + 4^{(-3)\times(-6)\times\frac{1}{6}} - 3^{(-4)\times\left(-\frac{1}{2}\right)}}{2^{(-3)\times(-4)\times\frac{1}{4}}}$$

$$\left[\because \left[\left(a^p\right)^q\right]^r = a^{pqr}\right]$$

$$= \frac{3^2 + 4^3 - 3^2}{2^3} = \frac{9 + 64 - 9}{8} = \frac{64}{8} = 8$$

$\therefore$ Expression = Part I + Part II = 2.4 + 8 = 10.4.

3. Find the value of $\sqrt{11 - 2\sqrt{30}} + \sqrt{7 - 2\sqrt{10}} - \frac{4}{\sqrt{6} + \sqrt{2}}$

[SSC Graduate Level (U DC) Main Exam 27.06. 2004]

Solution : Let Part I $= \sqrt{11 - 2\sqrt{30}}$

$= \sqrt{11 - 2\sqrt{5 \times 6}} = \sqrt{11 - 2 \times \sqrt{5} \times \sqrt{6}}$

$= \sqrt{6 + 5 - 2 \times \sqrt{5} \times \sqrt{6}}$

$= \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{5}\right)^2 - 2 \times \sqrt{5} \times \sqrt{6}}$

$= \sqrt{\left(\sqrt{6} - \sqrt{5}\right)^2} = \sqrt{6} - \sqrt{5}$

Part II $= \sqrt{7 - 2\sqrt{10}}$

$= \sqrt{7 - 2\sqrt{5 \times 2}} = \sqrt{5 + 2 - 2\sqrt{5} \times \sqrt{2}}$

$= \sqrt{\left(\sqrt{5}\right)^2 + \left(\sqrt{2}\right)^2 - 2 \times \sqrt{5} \times \sqrt{2}}$

$= \sqrt{\left(\sqrt{5} - \sqrt{2}\right)^2} = \sqrt{5} - \sqrt{2}$

Part III $= \frac{4}{\sqrt{6} + \sqrt{2}}$

On rationalising the denominator, we get,

$$= \frac{4}{\left(\sqrt{6} + \sqrt{2}\right)} \times \frac{\left(\sqrt{6} - \sqrt{2}\right)}{\left(\sqrt{6} - \sqrt{2}\right)}$$

$$= \frac{4\left(\sqrt{6} - \sqrt{2}\right)}{\left(\sqrt{6}\right)^2 - \left(\sqrt{2}\right)^2} = \frac{4\left(\sqrt{6} - \sqrt{2}\right)}{6 - 2} = \sqrt{6} - \sqrt{2}$$

$\therefore$ Expression = Part I + Part II - Part III

$= \left(\sqrt{6} - \sqrt{5}\right) + \left(\sqrt{5} - \sqrt{2}\right) - \left(\sqrt{6} - \sqrt{2}\right)$

$= \sqrt{6} - \sqrt{5} + \sqrt{5} - \sqrt{2} - \sqrt{6} + \sqrt{2} = 0$

4. Simplify : $\frac{2 + \sqrt{3}}{2 - \sqrt{3}}$

[SSC Graduate Level (U DC) Main Exam, 21.09.2003]

Solution : The given expression

$$= \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{\left(2 + \sqrt{3}\right)\left(2 + \sqrt{3}\right)}{\left(2 - \sqrt{3}\right)\left(2 + \sqrt{3}\right)}$$

$$= \frac{\left(2 + \sqrt{3}\right)^2}{(2)^2 - \left(\sqrt{3}\right)^2} = \frac{\left(2 + \sqrt{3}\right)^2}{4 - 3}$$

$= 4 + 3 + 2 \times 2 \times \sqrt{3} = 7 + 4\sqrt{3}$

5. If $x = \frac{\sqrt{3}}{2}$, find the value of $\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$

[SSC Graduate Level (Asstt. Grade) Main Exam. 29.12. 2002]

Solution :

$$\text{Expression} = \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{\left(\sqrt{1+x} + \sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2} = \frac{2 + 2\sqrt{1-x^2}}{2x}$$

$$= \frac{1 + \sqrt{1-x^2}}{x}$$

Putting $x = \frac{\sqrt{3}}{2}$,

$$\text{Expression} = \frac{1 + \sqrt{1 - \frac{3}{4}}}{\frac{\sqrt{3}}{2}} = \frac{1 + \frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{3}{2} \times \frac{2}{\sqrt{3}} = \sqrt{3}$$

6. Find the value of $16^{0.16} \times 2^{0.36}$

[SSC Graduate Level (U DC) Main Exam, 21.09.2003]

Solution : $16^{0.16} \times 2^{0.36}$

$= (2^4)^{0.16} \times 2^{0.36}$

$= 2^{0.64} \times 2^{0.36}$

$= 2^{0.64+0.36} = 2^1 = 2$

7. If $5^{\sqrt{x}} + 12^{\sqrt{x}} = 13^{\sqrt{x}}$, then x is equal to

(1) $\frac{25}{4}$ (2) 4

(3) 9 (4) 16

[SSS Graduate Level Prelim. Exam; 27.07.2008 (1st Sitting)]

Sol. (2) $5^{\sqrt{x}} + 12^{\sqrt{x}} = 13^{\sqrt{x}}$

We know that $5^2 + 12^2 = 13^2$

$\therefore \sqrt{x} = 2 \Rightarrow x = 2^2 = 4$

8. $\frac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to

(1) $1-\sqrt{5}+\sqrt{2}+\sqrt{10}$ (2) $1+\sqrt{5}+\sqrt{2}-\sqrt{10}$

(3) $1+\sqrt{5}-\sqrt{2}+\sqrt{10}$ (4) $1-\sqrt{5}-\sqrt{2}+\sqrt{10}$

[SSS Graduate Level Prelim. Exam; 04.02.2007 (1st Sitting)]

Sol. (2) Expression $= \frac{12}{3+\sqrt{5}+2\sqrt{2}}$

$= \frac{12(3+\sqrt{5}-2\sqrt{2})}{\left[(3+\sqrt{5})+2\sqrt{2}\right]\left[(3+\sqrt{5})-2\sqrt{2}\right]}$

[Rationalising the denominator]

$= \frac{12(3+\sqrt{5}-2\sqrt{2})}{(3+\sqrt{5})^2-(2\sqrt{2})^2} = \frac{12(3+\sqrt{5}-2\sqrt{2})}{9+5+6\sqrt{5}-8}$

$= \frac{12(3+\sqrt{5}-2\sqrt{2})}{6\sqrt{5}+6} = \frac{2(3+\sqrt{5}-2\sqrt{2})}{\sqrt{5}+1}$

$= \frac{2(3+\sqrt{5}-2\sqrt{2})(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)}$

$= \frac{2(3\sqrt{5}+5-2\sqrt{10}-3-\sqrt{5}+2\sqrt{2})}{5-1}$

$= \frac{2(2\sqrt{5}+2\sqrt{2}-2\sqrt{10}+2)}{4} = \frac{2\times 2(\sqrt{5}+\sqrt{2}-\sqrt{10}+1)}{4}$

$= 1+\sqrt{5}+\sqrt{2}-\sqrt{10}$

9. If $2^{2x-y} = 16$ and $2^{x+y} = 32$, the value of xy is

(1) 2 (2) 4

(3) 6 (4) 8

[SSC CPO Sub-Inspector Exam; 06.09.2009]

Sol. (3) $2^{2x-y} = 16 = 2^4$

$\Rightarrow 2x - y = 4$ .... (i)

$2^{x+y} = 32 = 2^5$

$\Rightarrow x + y = 5$ .... (ii)

On adding equations (i) and (ii),

$3x = 9 \Rightarrow x = 3$

From equation (ii),

$y = 5 - x = 5 - 3 = 2$

$\therefore xy = 3 \times 2 = 6$

10. $\frac{2}{\sqrt{5}+\sqrt{3}} - \frac{3}{\sqrt{6}-\sqrt{3}} + \frac{1}{\sqrt{6}+\sqrt{5}})$ is equal to

(1) $2\sqrt{6}$ (2) $2\sqrt{5}$

(3) $2\sqrt{3}$ (4) 0

[SSC CPO Sub-Inspector Exam; 09.11.2008]

Sol. (1) $\frac{2}{\sqrt{5}+\sqrt{3}} = \frac{2(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$

(rationalising the denominator)

$= \frac{2(\sqrt{5}-\sqrt{3})}{5-3} = \sqrt{5}-\sqrt{3}$

Similarly,

$\frac{3}{\sqrt{6}-\sqrt{3}} = \frac{3(\sqrt{6}+\sqrt{3})}{6-3} = \sqrt{6}+\sqrt{3}$

$\frac{1}{\sqrt{6}+\sqrt{5}} = \frac{\sqrt{6}-\sqrt{5}}{6-5} = \sqrt{6}-\sqrt{5}$

$\therefore$ Expression

$= \sqrt{5}-\sqrt{3}+\sqrt{6}+\sqrt{3}+\sqrt{6}-\sqrt{5} = 2\sqrt{6}$

11. If $a = \frac{\sqrt{5}+1}{\sqrt{5}-1}$ and $b = \frac{\sqrt{5}-1}{\sqrt{5}+1}$, the value of $\left(\frac{a^2+ab+b^2}{a^2-ab+b^2}\right)$ is

[SSC Section Officer (Commercial Audit) Exam; 30.09.2007]

(1) $\frac{3}{4}$ (2) $\frac{4}{3}$

(3) $\frac{3}{5}$ (4) $\frac{5}{3}$

Sol. (2) It is given that

$a = \frac{\sqrt{5}+1}{\sqrt{5}-1}$ and $b = \frac{\sqrt{5}-1}{\sqrt{5}+1}$

Now, $a + b = \frac{\sqrt{5}+1}{\sqrt{5}-1} + \frac{\sqrt{5}-1}{\sqrt{5}+1}$

$= \frac{(\sqrt{5}+1)^2+(\sqrt{5}-1)^2}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2\left[(\sqrt{5})^2+(1)^2\right]}{(\sqrt{5})^2-(1)^2}$

$[\because (a+b)^2 + (a-b)^2 = 2(a^2+b^2)]$

$= \frac{2(5+1)}{5-1} = \frac{2\times 6}{4} = 3$

and $a.\,b = \frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}-1}{\sqrt{5}+1} = 1$

Expression $= \frac{a^2+ab+b^2}{a^2-ab+b^2}$

$= \frac{(a+b)^2-ab}{(a+b)^2-3ab} = \frac{(3)^2-1}{(3)^2-3\times 1} = \frac{9-1}{9-3} = \frac{8}{6} = \frac{4}{3}$